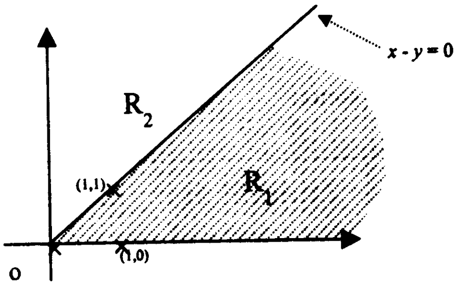
Choose a point say (5, 0) lies in the region R_1. Then substituting in the inequality $x + y \leq 10$, we have $5 + 0 \leq 10$. Hence (5, 0) satisfies the inequality, therefore R_1 is the required region.

To find the region of the constrains $x - y \geq 0$

$x - y = 0$:

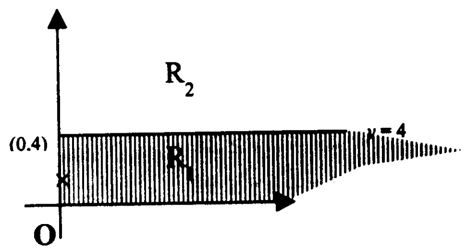
x	0	1
y	0	1



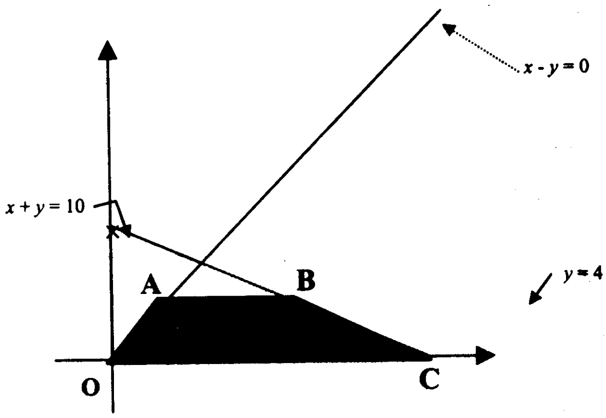
Choose a point say (1, 0) lies in the region R_1. Then substituting in the inequality $x - y \geq 0$, we have $1 - 0 \geq 0$. Hence (1, 0) satisfies the inequality, therefore R_1 is the required region.

To find the region of the constrains $y \leq 4$

The equation $y = 4$ represents a line parallel to x – axis through (0, 4)



On supper imposition of all the above three regions we get the following. Shaded region is the feasible region that is the intersection of all the regions.



To find corners of the feasible region OABC

O = (0, 0); C = (10, 0)

A is the intersection of

$x - y = 0$

$y = 4$

Solving we get : $x = 4, y = 4$

$\therefore$ A = (4, 4)

B is the intersection of

$x + y = 10$

$y = 4$

Solving we get : $x = 6, y = 4$

$\therefore$ B = (6, 4)

To find the optimal solution

$z = x + 2y$

Z at O = (0, 0)	Z at A = (4, 4)	Z at B = (6, 4)	Z at C = (10, 0)
$z = 0 + 0 = 0$	$z = 4 + 2(4) = 12$	$z = 6 + 2(4) = 14$	$z = 10 + 2(0) = 10$

$\therefore$ Max z = Max {0, 12, 14, 10} = 14, which is at B = (6, 4).

Hence maximal value of z = 14 (optimal value).

Optimal solution is (6, 4) $\Rightarrow x = 6,\ y = 4$.

Hence to get a maximum pleasure of 14 units Kiran has to work for 6 hours and play for 8 hours.

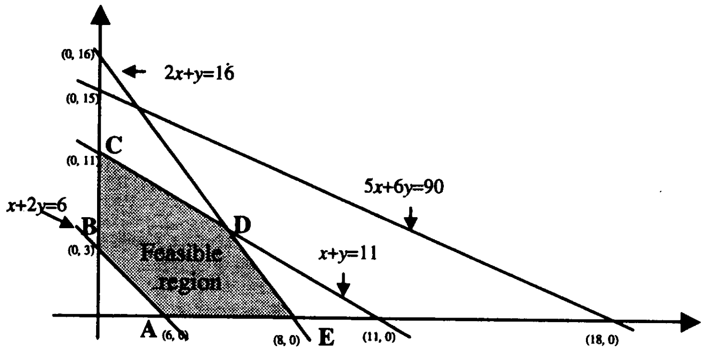
Example 2: Use graphical method to solve the L.P.P.

$$\textbf{Max } z = 120x + 100y$$

Subject to constraints

$$10x + 5y \le 80$$
$$6x + 6y \le 66$$
$$4x + 8y \ge 24$$
$$5x + 6y \le 90$$
$$x \ge 0, y \ge 0$$

Solution: Graph for the constraints is



Feasible region is ABCDE

Corner point	A = (6, 0)	B = (0, 3)	C = (0, 11)	D = (5, 6)	E = (8, 0)
Value of $z = 120\,x + 100\,y$	720	300	1100	1200	960

Optimum value of $z = 1200$; Optimal solution is $x = 5, y = 6$

Example 3: Two grades of paper *M* and *N* are produced on a paper machine. Because of raw material restriction not more than 400 tones of grade *M* and 300 tones of grade *N* can be produced in a week. It requires 0.2 and 0.4 hours to produce a tone of products *M* and *N* respectively, with corresponding profits of Rs. 20 and Rs. 40 per tone. It is given that there are 160 hours in a week.

Formulate the problem as a L.P.P. and determine the optimum product mix.

Solution: Data table:

Profit in Rs.	Product	Time requirement in hrs.	Availability in tones per week.
20	*M*	0.2	400
40	*N*	0.4	300
Time bound per week		160	

Let x and y be respectively the number of tones of production of M and N per week.

Mathematical model is

Objective function

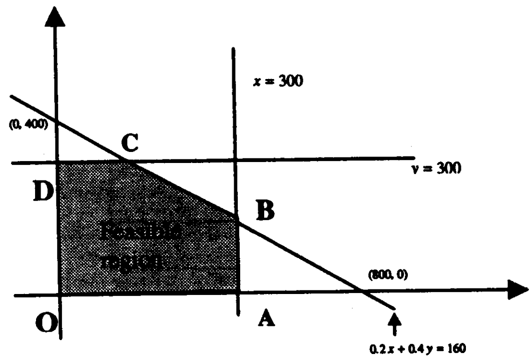
$$\text{Max } z = 20\,x + 40\,y$$

Subject to the constraints

$$0.2\,x + 0.4\,y \leq 160$$

$$0 \leq x \leq 400,\ 0 \leq y \leq 300$$

The graph of the constraints is



Feasible region is OABCD

Corner point	O = (0, 0)	A = (400, 0)	B = (400, 200)	C = (200, 300)	D = (0, 300)
$z = 20x + 40y$	0	8000	16000	16000	12000

The maximum value of $z = 16000$

Optimal solutions are C and B. Since the model contains two optimal solutions, the set of all points lies in the line $0.2x + 0.4y = 160$ between B and C are the optimal solutions to the problem. Hence the problem contains infinite number of solutions. (observe (300, 250) lies in the line and $z = 16000$ at this point also).

Example 4: Show graphically that the L.P.P.

$$\text{Max } z = 6x_1 + x_2$$

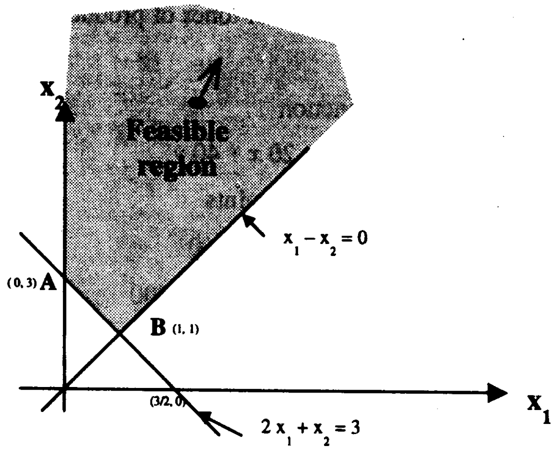
Subject to the constraints

$$2x_1 + x_2 \geq 3$$

$$x_1 - x_2 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0 \text{ has an unbounded solution.}$$

Solution: The graph of the constraints is



Feasible region is an unbounded convex region.

Corner point	Value of z
A = (0, 3)	3
B = (1, 1)	7

Optimal value of $z = 7$, at (1, 1). Now consider any point say (10, 12), the value of z at (10, 12) is 72 which is greater than the optimal value 7. Hence the maximum flow is always moving from (1, 1) towards the point (10, 12). Hence the given problem has an unbounded solution (i.e. the maximum value of z attains at $x = \infty$ and/or $y = \infty$).

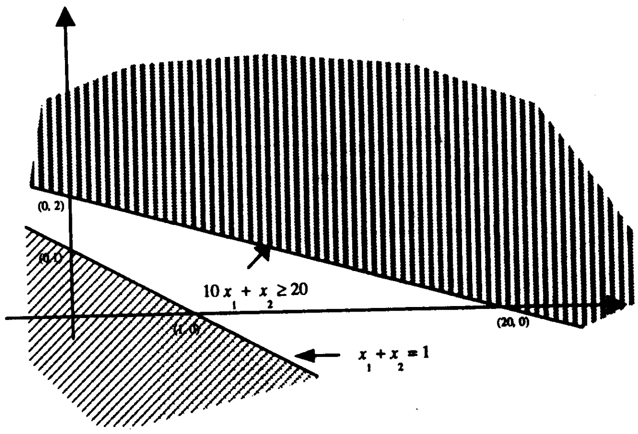
Example 5. Show graphically that the model

$$\text{Max } Z = -5x_2$$

Subject to

$x_1 + x_2 \le 1 : -0.5x_1 - 5x_2 \le -10 : x_1 \ge 0, x_2 \ge 0$ has no feasible solution.

Solution: The graph of the constraints is



From the graph it is clear that the region of intersection of the constrains is empty. Hence the given problem contains no feasible region (solution). Therefore the given problem has no solution.

Solution of a minimization model:

The general form and steps in the solution of the problem are similar to that of maximization problem. The objective function is to Minimize. Hence in the last step choose the corner point among all the corner points, which gives the minimum value for Z.

Example 6. Mr. Kumar uses at least 800 lbs of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions

Feed stuff	lb per lb of feed stuff		
	Protein	Fiber	Cost in Rs. per lb.
Corn	0.09	0.02	0.30
Soybean	0.60	0.06	0.90

The dietary requirement of the special feed stipulate at least 30% protein and at most 5% fiber. Kiran wishes to determine the daily minimum cost feed mix. Solve the problem graphically.

Solution: Let x and y be respectively the weights of corn and soybean to be taken. Then the mathematical model for the problem is

Min $z = 0.30\,x + 0.90\,y$

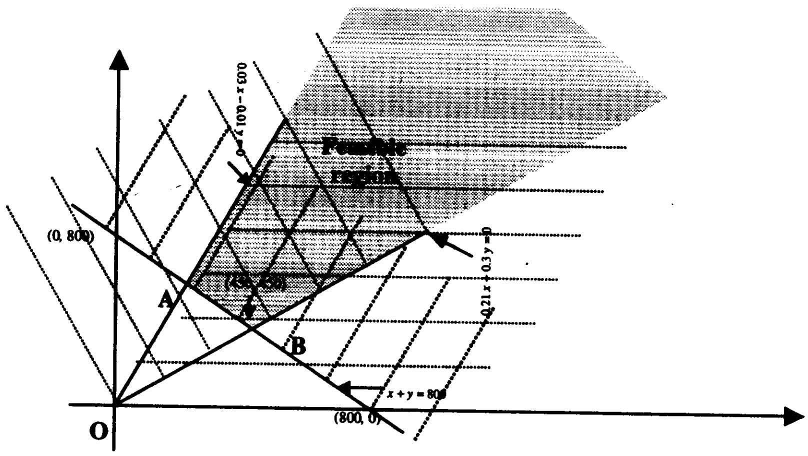
Subject to

$0.09\,x + 0.60\,y \geq 30\%$ of total $= 0.30\,(x + y) \;\Rightarrow 0.21\,x - 0.3\,y \leq 0$

$0.02\,x + 0.06\,y \leq 5\%$ of total $= 0.05\,(x + y) \;\Rightarrow 0.03\,x - 0.01\,y \geq 0$

$x + y \geq 800$

The graph of the constraints is



Corner point	A = (400, 400)	B = (470.6, 329.4)
$z = 0.3\,x + 0.9\,y$	480	437.64

The minimum z = 437.64 at B. Since the region is unbounded, consider a point in the feasible region say (450, 450), the value of z is 540 that is greater than the minimum value. Hence the flow is towards the point B. Thus B is the optimal solution.

Therefore in order to spend a minimum cost of Rs. 437.64 satisfying all the requirements Kumar has to consume 470.6 lbs of Corn and 329.4 lb of Soybean.

Example 7: A person requires 10, 12 and 12 units of vitamins *A*, B and C respectively. A liquid product contains 5, 2 and 1 units of *A*, *B* and *C* respectively per jar. A dry product contains 1, 2 and 4 units of *A*, *B* and *C* per carton. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per carton, how many of each should he purchase in order to minimize the cost and meet the requirement?

Solution:

Product	Contents			Price in Rs.
	A	*B*	*C*	
Liquid	5	2	1	3
Dry	1	2	4	2
Requirement	10	12	12	

Let x and y be denote respectively amounts of liquid jars and dry cartons to be purchased. Then cost $Z = 3x + 2y$

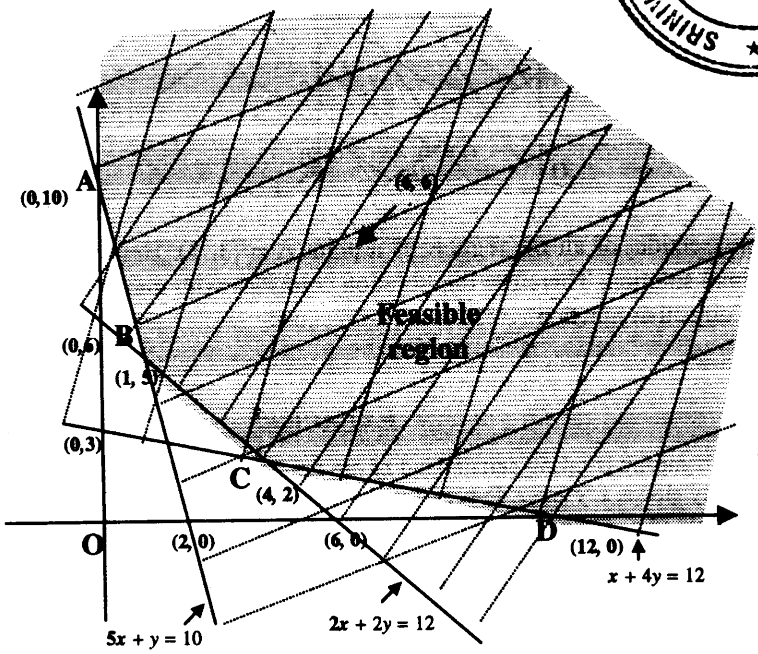
Objective function

Minimize $Z = 3x + 2y$

Subject to the constraints

$$5x + y \geq 10$$
$$2x + 2y \geq 12$$
$$x + 4y \geq 12$$
$$x \geq 0,\ y \geq 0$$

The graph of the constraints is



The feasible region is an unbounded convex set.

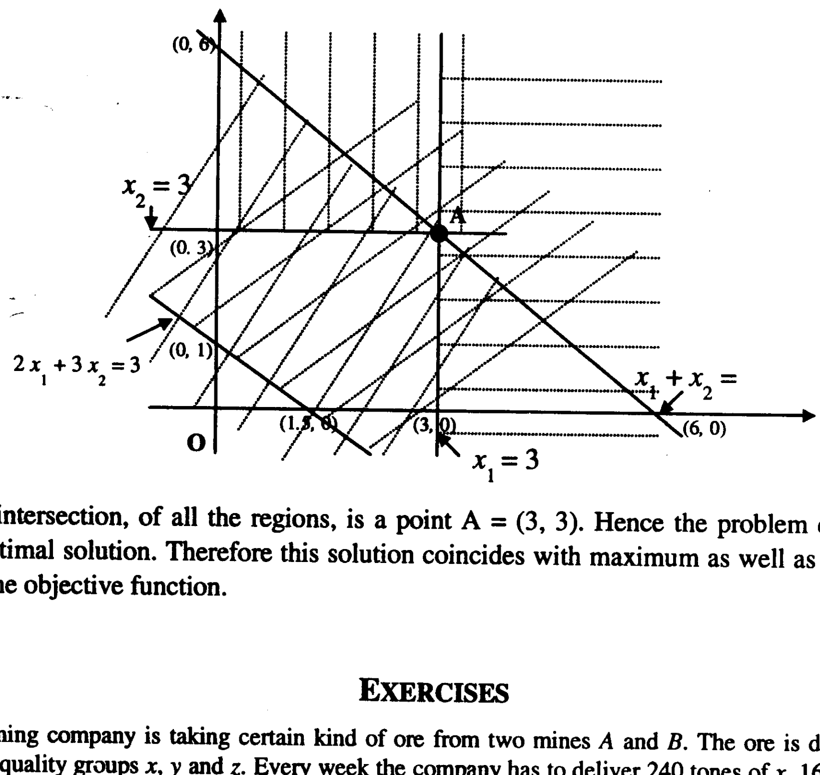
Corner point	A = (0, 10)	B = (1, 5)	C = (4, 2)	D = (12, 0)
$Z = 3x + 2y$	20	13	16	36

The minimum $Z = 13$ at B = (1, 5). Consider any point say (6, 6) in the feasible region, then Z at (6, 6) = 30 > 13. Hence minimum falls towards B. Therefore optimum value is 13; optimal solution is $x = 1$ and $y = 5$.

Conclusion: The person has to take one jar of liquid product and 5 cartons of dry product in order to spend a minimum of Rs. 13 to meet all of his requirements.

Example 8. Show graphically that the maximum or minimum value of the objective function $Z = 5x_1 + 3x_2$, subject to constraint $x_1 + x_2 \leq 6$; $2x_1 + 3x_2 \geq 3$; $0 \leq x_1 \geq 3$; $0 \leq x_2 \geq 3$ is same.

Solution: The graph of the constraints is



The intersection, of all the regions, is a point A = (3, 3). Hence the problem contains a unique optimal solution. Therefore this solution coincides with maximum as well as minimum value of the objective function.

EXERCISES

1. A mining company is taking certain kind of ore from two mines *A* and *B*. The ore is divided into three quality groups *x*, *y* and *z*. Every week the company has to deliver 240 tones of *x*, 160 tons of *y* and 440 tones of *z*. The cost per day for running mine *A* is Rs 3,000 and for running mine *B* is Rs 2,000. Each day *A* will produce 60 tons of *x*, 20 tones of *y* and 40 tons of *z*, while *B* will produce 20 tons of *x*, 20 tons of *y* and 80 tons of *z*. Using graphical method, find the most economical production plan.

2. The constraints of a linear program are as follows:

$$3x_1 + x_2 \geq 8 \, ; 2x_1 + 5x_2 \geq 14 \ ; 3x_1 - x_2 \leq 36$$
$$6x_1 + x_2 \leq 99 \, ; 0 \leq x_2 \leq 15, x_1 \geq 0.$$

   Its objective function is given by $f(x_1, x_2) = 6x_1 + x_2$.

   Find graphically, the maximum and minimum values of the program and corresponding solutions. Also mark on the graph the set of feasible solutions at which the value of the objective function is 30.

3. Find the minimum value of $Z = 20x_1 + 10x_2$

   Subject to constraints:

$$x_1 + x_2 \leq 40; \ 3x_1 + x_2 \geq 30; \ 4x_1 + 3x_2 \geq 60; \ x_1, x_2 \geq 0.$$

4. Find the minimum value of $z = 2x + 3y$

Subject to constraints:

$$x + y \leq 30;\; x - y \geq 0,\; y \geq 3;\; 0 \leq x \leq 20;\; 0 \leq y \leq 12.$$

5. Maximize $z = 3x_1 + 4x_2$

Subject to the constraints:

$$3x_2 - x_1 \leq 3;\; x_2 - x_1 \leq 0;\; x_2 \geq 1, x_2 \geq 0$$

6. Old hens can be fought at Rs. 2 each and young ones at Rs. 5 each. The old hens can lay 3 eggs per week and the young one lay 5 eggs per week, each egg being worth 30 paise. A hen cost Rs. 1 per week to feed. Vinod has only Rs. 82 spend for hens. How many each kind should Vinod buy to give a profit of more than Rs. 6 per week, Assuming that Vinod cannot house more than 20 hens. Solve the linear programming problem graphically.

7. Maximize $z = 2x_1 + 3x_2$

Subject to the constraints

$$x_1 + x_2 \geq 1;\; x_1 - 5x_2 \leq 0;\; 5x_1 - x_2 \geq 0;\; x_2 - x_1 \geq -1$$

$$x_1 + x_2 \leq 6;\; x_2 \leq 3, x_1 \geq 0, x_2 \geq 0$$

8. The standard weight of special purpose brick is 5 kg. and it contains two basic ingredients B_1 and B_2. B_1 cost Rs. 5 per kg. and B_2 cost Rs. 8 kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and minimum of 2 kg. of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically the minimum cost of the brick satisfying the above condition.

9. 10 grams of alloy *A* contains 2 grams of Copper, 1 gram of Zinc and 1 gram of Lead. 10 grams of alloy *B* contains 1 gram of Copper, 1 gram of Zinc and 3 grams of Lead. It is required to produce a mixture of these alloys that contains at least 10 grams of copper, 8 grams of Zinc and 12 grams of Lead. Alloy *B* contains 15 times as much per kgs as alloy *A*. Find the amounts of alloy *A* and alloy *B* that must be mixed in order to satisfy these conditions in the cheapest way.

10. A firm manufacturers two products *A* and *B* on which the profit earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product *A* requires one minute of processing time on M_1 and two minutes on M_2, while *B* requires 1 minute on M_1 and one minute on M_2. Machine M_1 is available for not more that 7 hours and 30 minuets, while machine M_2 is available for 10 hours during any working days. Find the number of units of products *A* and *B* to be manufactured to get maximum profit.

Answers:

(1) Optimal solution is (14, 42); Optimal value is 1,26,000.
(2) Maximum value is 99 at any point in the line segment between the points (14, 15) and (15, 9). Minimum value is 8 at (0, 8).
(3) Minimum value is 240 at (6, 12).
(4) Optimal value is 72; solution is (18, 12).
(5) Maximum value is 17 at (4, 3)
(6) Only 16 young hens to get a maximum profit of Rs. 8.
(7) Maximum is 17 at (4, 3)
(8) Optimal value is 31 at (3, 2).
(9) Produce only alloy *A* 120 gm for minimum cost Rs. 0.12.
(10) Maximum profit Rs. 1,800 at (0, 450)

2.4 Simplex method

The graphical method can be applied to solve LP model involving two decision variables. We know that the optimal solution is always associated with the corner points of the feasible solution. Using this key idea, the simplex methods to solve LP model involving two or more variables are developed.

The standard LP form:

Objective function

Optimize $z = f(x_1, x_2, \ldots, x_n)$

Subject to the constraints

$g_i(x_1, x_2, \ldots, x_n) = b_i$, $b_i \geq 0$, $i = 1, 2, \ldots, m$. $m < n$.

$x_j \geq 0$, for all $j = 1, 2, \ldots, n$

where $x_1, x_2, \ldots, x_n$ are decision variables.

Thus in standard form of an LPP

(i) The objective function may be of the maximization or minimization type.

(ii) All the constraints are (except non-negative restriction) are equations with non-negative right hand side.

(iii) All the variables are non-negative.

How to convert LP into standard form?

1. The objective function can be converted into maximization type from minimization type and vice versa as:

$$\text{Min } z = -\text{Max}(-z)$$

For example, the objective function

$$\text{Min } z = 2x_1 + 3x_2 - 6x_3 + 7x_4 \text{ can be written as}$$

$$-\text{Max}(-z) = -\text{Max}\{-2x_1 - 3x_2 + 6x_3 - 7x_4\}.$$

2. If right hand side of any inequality in the constraint is greater than the left hand side, then the inequality can be made equation by adding a non-negative variable s_1 to left hand side and assigning a zero cost vector for s_1 in the objective function.

For example, $2x_1 + x_2 \leq 3$ can be written as $2x_1 + x_2 + s_1 = 3$, $s_1 \geq 0$.

Such a variable s_1, we add in the left hand side of the constraint to convert into equation is called **slack** variable. A slack thus represents the amount by which the available amount of the resource exceeds its usage by the activity.

3. If the right hand side of any inequality in the constraint is lesser than the left hand side, then the inequality can be made equation by subtracting a non-negative

variable s_1 form the left hand side and assigning a zero cost vector for s_1 in the objective function.

For example, $2x_1 + x_2 \geq 3$ can be written as $2x_1 + x_2 - s_1 = 3, s_1 \geq 0$.

Such a variable s_1, we are subtracting from the left hand side to convert the constraint into equation is called **surplus** variable. A surplus thus represents the excess of the left hand side over the minimum requirement.

4. Any unrestricted variable x_i can be written as

$$x_i = x_i^+ - x_i^-, \text{ where } x_i^+ \geq 0, x_i^- \geq 0$$

For example, if $x_3 = 5$, then x_3 can be written as $x_3 = 8 - 3$, or if $x_3 = -5$, then $x_3 = 3 - 5$.

Example 1. Express the following L.P.P. in standard form

$$\text{Max } Z = 2x_1 + 3x_2 + 5x_3$$

Subject to

$$x_1 + x_2 - x_3 \geq 5$$
$$-6x_1 + 7x_2 - 9x_3 \leq 4$$
$$x_1 - x_2 + 3x_3 \leq -5$$
$$x_1 + x_2 + 4x_3 = 10$$
$$x_1, x_2 \geq 0.$$

Solution: x_3 is the unrestricted variable thus: $x_3 = x_3^+ - x_3^-$ where $x_3^+, x_3^- \geq 0$.

First constraint is

$$x_1 + x_3 - x_3 \geq 5$$
$$x_1 + x_2 - x_3 - s_1 = 5, \quad s_1 \text{ (is a surplus variable)} \geq 0$$

Second constraint is

$$-6x_1 + 7x_2 - 9x_3 \leq 4$$
$$-6x_1 + 7x_2 - 9x_3 + s_2 = 4, \quad s_2 \text{ (is a slack variable)} \geq 0$$

Third constraint is

$$x_1 - x_2 - 3x_2 \leq -5$$
$$-x_1 + x_2 - 3x_2 \geq 5 \quad \text{(converting RHS into positive member).}$$
$$-x_1 + x_2 - 3x_2 - s_3 = 5, \quad s_3 \text{(is a surplus variable)} \geq 0.$$

Thus, the given L.P.P. can be written in the standard form as

$$\text{Max } Z = 2x_1 + 3x_2 + 5\left(x_3^+ - x_3^-\right) + 0\,s_1 + 0\,s_2 + 0\,s_3.$$

Subject to

$$x_1 + x_2 - \left(x_3^+ - x_3^-\right) - s_1 = 5$$

$$-6x_1 + 7x_2 - 9\left(x_3^+ - x_3^-\right) + s_2 = 4$$

$$-x_1 + x_2 - 3\left(x_3^+ - x_3^-\right) - s_3 = 5$$

$$x_1 \geq 0, x_2 \geq 0, x_3^+ \geq 0, x_3^- \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$$

2.5 Basic Solution

The standard form of a given L.P.P. includes m simultaneous linear equation in n- unknown or variables ($m < n$). Equating $n - 1$ variables zero and solving the m equation in m variables, we get a unique solution. This solution is called **basic solution**, the associated variables are called **basic variables** and those variables equated to zero are called **non-basic variables.**

Example 2 : Find a set of basic solution for the following constraints

$$x_1 + x_2 - x_3^+ + x_3^- - s_1 = 5$$

$$-6x_1 + 7x_2 - 9x_3^+ + 9x_3^- - s_2 = 4$$

$$-x_1 + x_2 - 3x_3^+ + 3x_3^- - s_3 = 5$$

Solution: Here there are 3 equations in 7 variables namely, x_1, x_2, x_3^+, x_3^- s_1, s_2 and s_3.

Taking any 3 variables say x_1, x_2, x_3^+ and equating remaining $7 - 3 = 4$ variables to zero we get

$$x_1 + x_2 - x_3^+ = 5$$

$$-6x_1 + 7x_2 - 9x_3^+ = 4$$

$$-x_1 + x_2 - 3x_3^+ = 5$$

Solving these three simultaneous equations we get,

$x_1 = 31/11, x_2 = -7/11$ and $x_3^+ = -31/11$ as a basic solution.

Basic variables are x_1, x_2, x_3^+, and non-basic variables are x_3^-, s_1, s_2 and s_3.

2. 6 Basic feasible solution

The basic solutions are classified into four categories depending upon the consistency* of the m-simultaneous equations in m basic variables obtained by equating non-basic variables to zero. If the system has unique solution (basic solution) and all the basic variables satisfy non-negativity restriction (i.e. all are grater than or equal to zero) then the basic solution is called **basic feasible solution.** Otherwise if any basic variable is negative in the basic solution then the basic solution is **called basic infeasible solution.**

* Recall consistency of system of equations

If the system is inconsistent then it contains no basic solutions corresponds to the basic variables chosen. Hence it is the case of non-existing basic solution. Otherwise the system is consistent and all the equations are not linearly independent (rank of coefficient matrix $< m$) then the L.P.P. contains infinite number of basic solutions.

Example 3. Determine all the possible basic solutions of the following L.P.P. and hence obtain its optimal solution.

$$\text{Max } Z = 2x_1 - 4x_2$$

Subject to

$$x_1 + 4x_2 \le 2$$
$$-x_1 + 3x_2 \le 1$$
$$x_1, x_2 \ge 0$$

Solution: The standard form of L.P.P. is

$$\text{Max } Z = 2x_1 - 4x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$\left.\begin{aligned} x_1 + 4x_2 + s_1 &= 2 \\ -x_1 + 3x_2 + s_2 &= 1 \\ x_1 \ge 0, x_2 \ge 0, s_1 &\ge 0, s_2 \ge 0. \end{aligned}\right\} \text{---------}(\ast)$$

There are 4 variables and 2 equations.

We can choose any 2 basic variables out of 4 in ${}^4C_2 = 6$ ways.

(i) Choosing x_1 and x_2 basic variables (i.e. $s_1 = s_2 = 0$)

From $(\ast)$ we get

$$x_1 + 4x_2 + 0 = 2$$
$$-x_1 + 3x_2 + 0 = 1$$

Solving we get: $x_2 = 3/7, x_1 = 2/7$ is a basic feasible solution

(ii) Choosing x_1 and s_1 basic variables:

From $(\ast)$ we get

$$x_1 + s_1 = 2$$
$$-x_1 + 0 + 0 = 1$$

$\Rightarrow$ $x_1 = -1, s_1 = 3$ is a basic infeasible solution.

(iii) Choosing x_1 and s_2 basic variables:

From $(\ast)$ we get

$$x_1 + 0 = 2$$
$$-x_1 + s_2 = 1$$

$\Rightarrow$ $x_1 = 2, s_2 = 3$ is a basic feasible solution.

(iv) Choosing x_2 and s_1 basic variables:

From (•) we get

$$x_2 + s_1 = 2$$
$$0 + 3x_2 = 1$$

$\Rightarrow$ $x_2 = 1/3,\ s_1 = 5/3$ is a basic feasible solution.

(v) Choosing x_2 and s_2 basic variables:

From (•) we get

$$4x_2 = 2$$
$$3x_2 + s_2 = 1$$

$\Rightarrow$ $x_2 = 1/2,\ s_2 = -1/2$ is a basic infeasible solution.

(vi) Choosing s_1 and s_2 basic variables:

From (•) we get

$$0 + s_1 = 2$$
$$0 + s_2 = 1$$

$\Rightarrow$ $s_1 = 2,\ s_2 = 1$ is a basic feasible solution.

$\therefore$ *Basic feasible solutions are*;

(i) $x_1 = 2/7, x_2 = 3/7, s_1 = 0, s_2 = 0$; The value of z corresponding to this solution is

$$z = 2(x_1) - 4(x_2) + 0\,s_1 + 0\,s_2 = 2(2/7) - 4(3/7) + 0 + 0 = -8/7$$

(ii) $x_1 = 2, x_2 = 0, s_1 = 3, s_2 = 3$; The value of z corresponding to this solution is

$$z = 2(2) - 4(0) + 0 + 0 = 4$$

(iii) $x_1 = 0, x_2 = 1/3, s_1 = 5/3, s_2 = 0$; The value of z corresponding to this solution is

$$z = 2(0) - 4(1/3) + 0 + 0 = -4/3$$

(iv) $x_1 = 0, x_2 = 0,\ s_1 = 2,\ s_2 = 1$; The value of z corresponding to this solution is

$$z = 0 + 0 + 0 + 0 = 0$$

The Max $z = \text{Max}\{-8/7, 4, -4/3, 0\} = 4$ $\therefore$ Optimal value = 4

Optimal solution is $x_1 = 2,\ x_2 = 0$ (s_1 and s_2 slack variable can be neglected).

The basic solutions obtained above are clearly the corner points of the feasible solution. Reader is advice to verify the fact by using graphical method to solve the above L.P.P.

EXERCISES

1. Write down the following LP problems in the standard form. Also find set of all its basic feasible solutions.

(i) Maximize $z = 3x_1 + 2x_2$ subject to the constraints:

$$x_1 + x_2 \le 4$$
$$x_1 - x_2 \le 2$$
$$x_1 \ge 0, x_2 \ge 0$$

(ii) Maximize $z = 2x_1 + 3x_2$

Subject to the constraints

$$x_1 + x_2 \le 4$$
$$-x_1 + x_2 \le 1$$
$$x_1 + 2x_2 \le 5$$
$$x_1 \ge 0, x_2 \ge 0.$$

(iii) Maximize $z = 8x_1 - 2x_2$

Subject to the constraints

$$-4x_1 + 2x_2 \le 1$$
$$5x_1 - 4x_2 \le 3$$
$$x_1 \ge 0, x_2 \ge 0.$$

(iv) Minimize $z = x_1 + x_2 + 3x_3$

Subject to the constraints

$$3x_1 + 2x_2 + x_3 \le 3$$
$$2x_1 + x_2 + 2x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0.$$

(v) Maximize $z = 3x_1 + 2x_2 + 5x_3$

Subject to the constraints

$$x_1 + 2x_2 + x_3 \ge 4$$
$$2x_1 + 2x_3 \le 60$$
$$x_1 + 4x_2 \le 30$$
$$x_1 - x_2 + x_3 \le -2$$
$$x_1 \ge 0, x_2 \le 0 \text{ and } x_3 \text{ is unrestricted.}$$

2.7 Simplex Algorithm

In actual process we need to determine ${}^{n}C_{m}$ basic variables to find an optimal solution for a given LPP involving n decision variables with m constraints. This involves large computations in various steps. To avoid this we adopt the following procedure, which yield a better basic feasible solution from the older one. This is an interactive process and called **simple method.**

The following are the steps involved in the simplex method which involving only slack variable (i.e. all the constraints are of the type ≤ and the RHS is positive) in its standard form.

Simplex method – I

Step 1. Input/construct the simplex table by choosing the slack variables as the basic variables as;

	Cost vector→	C:	c_1	c_2		…		c_n	
Cost vector ↓	Basic variables ↓	RHS of the constraint ↓	variables						Intercept/Ratio $\frac{x_{B_i}}{y_{ij}}, y_{ij} > 0$ ↓
C_B	y_B	x_B	x_1	x_2	…	x_j	…	x_n	
C_{B1}		x_{B1}	y_{11}	y_{12}		y_{1j}		y_{1n}	
⋮		⋮							
						y_{ij} Pivot			Least ←positive*
⋮		⋮							
C_{Bn}		x_{Bn}	y_{m1}	y_{m2}		y_{mj}		y_{mn}	
	z_j	Z	z_1	z_2				z_n	
		Net evaluation $z_j - c_j$				↑ Most negative#			

Find value of the objective function $z = \sum_{i=1}^{m} C_{B_i} x_{Bi}$

---

* In case of tie to choose the least one, replace x_{Bi} by x_{ki} where k is the first column of the identity matrix corresponds to the basic variables if there is no tie again, otherwise go to next column of the identity matrix and so on, only for those rows where there is a tie. The situation is called degeneracy.

Choose most positive in the case of minimization problem

Step 2. Compute the net evaluation $z_j - c_j$ for each column j, $1 \le j \le n$, where $z_j = \sum_{i=1}^{m} c_{B_i} y_{ij}$.

If $z_j - c_j \ge 0$ for all $j = 1, 2, \ldots, n$ and the objective function is to maximize**, then stop, the optimal solution is reached. Otherwise, if $z_j - c_j < 0$, for any j, then choose the column j for which the net evaluation $z_j - c_j$ is most negative.

i.e. j is such that $z_j - c_j = \min_k \{z_k - c_k\}^*$.

Step 3. For the column j, chosen in step 2, compute the ratio (intercept) $\frac{x_{B_i}}{y_{ij}}$, $1 \le i \le m$, for which $y_{ij} > 0$. Choose the row i such that $\frac{x_{B_i}}{y_{ij}} = \min_{1 \le k \le m} \left\{ \frac{x_{B_k}}{y_{kj}} \right\}$

Step 4. choose the pivotal element y_{ij}, where j and i are chosen in step 2 and 3.

Step 5. Take the decision of leaving the i^{th} row.

Perform the Gaurs Jordan operation to convert j^{th} row as i^{th} row without affecting other rows corresponds to the basic variables. This can be achieved with the following operations.

(i) **For** the pivotal row (i^{th} row chosen in step 4)

$k = 1$

while $(k \le n)$

$$y_{ik} \leftarrow \frac{y_{ik}}{y_{ij}}, \; x_{B_i} \leftarrow \frac{x_{B_i}}{y_{ij}}$$

(ii) **For** each row l ($l \ne i$)

$$x_{B_l} \leftarrow x_{B_l} - \frac{x_{B_i}}{y_{ij}} \times y_{lk}$$

$$y_{lk} \leftarrow y_{lk} - \frac{y_{ik}}{y_{ij}} \times y_{lk}$$

Return to step 1

Example 1: Use simplex method to

Maximize $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$

Subject to the constraints:

$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$; $3x_1 + 3x_2 + x_3 + x_4 \le 80$; $2x_1 + 2x_3 + x_4 \le 60$;

$x_j \ge 0$, $j = 1, 2, 3, 4$

** Replace '≤' by '≥' in the statement of step 2, if the objective function is to minimize.

* Maximum in case of Minimization problem.

Solution: The given LP in standard form is

Maximize $z = 4x_1 + 3x_2 + 4x_3 + 6x_4 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 + s_1 = 80$$
$$3x_1 + 3x_2 + x_3 + x_4 + s_2 = 80$$
$$2x_1 + 2x_3 + x_4 + s_3 = 60$$

$x_j \geq 0, j = 1, 2, 3, 4$; and $s_1, s_2, s_3 \geq 0$ are slack variables.

Simplex table –1 (initial iteration)

		C:→	4	3	4	6	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
0	s_1	80	1	2	2	4	1	0	0	80/4 = 20 ←
0	s_2	80	3	3	1	1	0	1	0	80/1 = 80
0	s_3	60	2	0	2	1	0	0	1	60/1 = 60
	z_j	0	0	0	0	0	0	0	0	
		$z_j - c_j$	-4	-3	-4	-6 ↑	0	0	0	

Computing the net evaluation $z_j - c_j$ for each column j, $1 \leq j \leq 6$; we observe form the above table that the net evaluation is not all non-negative. Thus the solution is not optimal. Choose the column 4 headed by x_4, for which the net evaluation is most negative. Computing the ratio x_{Bi}/y_{i4} for each of positive entries in the column 4, we have the ration 20 is least positive and is in the row 1. Thus the row 1 to be chosen. The pivotal element is thus 4 being the intersection of the column 4 and the row 1.

The basic variable s_1 corresponding to the first row should leave the basis and the variable x_4 corresponding to the column 4 should enter the basis.

Performing Gauss Jordan operation;

(i) For the pivotal row 1

$k = 1$

while $(k \leq n)$

$$y_{ik} \leftarrow \frac{y_{ik}}{y_{ij}}, x_{B_i} \leftarrow \frac{x_{B_i}}{y_{ij}} \text{ with } i = 1, j = 4$$

i.e. $R_1 \leftarrow R_1 / y_{14} = R_1/\text{pivotal} = R_1/4$, we get

[redacted]

(iii) For each row l ($l \neq 1$)

$$x_{B_l} \leftarrow x_{B_l} - \frac{x_{B_l}}{y_{ij}} \times y_{lk}$$

$$y_{lk} \leftarrow y_{lk} - \frac{y_{1k}}{y_{14}} \times y_{lk}$$

i.e. $R_l = R_l - (R_1 / y_{14}) \times$ (pivotal element of the row l), $l = 2, 3$

[redacted]

The simplex table now reduces to the following:

Simplex table –2

		C:→	4	3	4	6	0	0	0	Ratio
c_B	y_B	x_B	[redacted]	x_2	x_3	x_4	s_1	s_2	s_3	
6	x_4	20	[redacted]	½	½	1	¼	0	0	80
0	s_2	[redacted]	[redacted]	[redacted]	[redacted]	[redacted]	[redacted]	[redacted]	[redacted]	[redacted]
0	s_3	40	[redacted]	- ½	3/2	0	- ¼	0	1	70
	z_j	120	[redacted]	3	3	6	3/2	0	0	
		$z_j - c_j$	[redacted]	0	-1	0	3/2	0	0	

From the table it is clear that the basic variable s_2 should leave the basis and the non-basic variable x_1 should enter the basic. Repeating the above steps this table we get the next iteration. The iteration is shown in the following table.

Simplex table –3

			$C:\rightarrow$	4	3	4	6	0	0	0	Ratio
	c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
$R_1 - \frac{R_2}{11}$	6	x_4	$\frac{160}{11}$	0	$\frac{3}{11}$	[redacted]	1	$\frac{3}{11}$	$-\frac{1}{11}$	0	32
$\frac{R_2 \times 4}{11}$	4	x_1	$\frac{210}{11}$	1	$\frac{10}{11}$	[redacted]	0	$-\frac{1}{11}$	$\frac{4}{11}$	0	120
$R_3 - \frac{R_2 \times 7}{11}$	0	s_3	[redacted]	[illegible]	[redacted]	[redacted]	[illegible]	[redacted]	[illegible]	[illegible]	[redacted]
		z_j	$\frac{1920}{11}$	4	$\frac{58}{11}$	[redacted]	6	$\frac{14}{11}$	$\frac{8}{11}$	0	
			$z_j - c_j$	0	$\frac{25}{11}$	[redacted]	0	$\frac{14}{11}$	0	0	

From the table it is clear that the basic variable s_3 should leave the basis and the non-basic variable x_3 should enter the basis. Thus we have the following table

Simplex table – 4

			$C:\rightarrow$	4	3	4	6	0	0	0
	c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
$R_1 - \frac{R_3 \times 5}{13}$	6	x_4	180/13	0	14/13	0	1	4/13	2/13	-5/13
$R_2 - \frac{R_3 \times 2}{13}$	4	x_1	280/13	1	16/13	0	0	- 1/13	6/13	-2/13
$\frac{R_3 \times 11}{13}$	4	x_3	20/13	0	-23/13	1	0	- 1/13	-7/13	11/13
		z_j	2280/11	4	56/13	4	6	16/13	8/13	6/13
			$z_j - c_j$	0	17/13	0	0	16/13	8/13	6/13

The net evaluation $z_j - c_j$ is non-negative for every column j. Hence the basic feasible solution is optimal basic feasible solution. The optimal value of the objective function is 2280/11 and the corresponding optimal solution is $x_1 = 280/13$, $x_2 = 0$, $x_3 = 20/13$ and $x_4 = 6$.

Example 2: Use simplex method to

$$\text{Minimize } z = 4x_1 - 3x_2 + 4x_3 - 6x_4$$

Subject to the constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$$
$$3x_1 + 3x_2 + x_3 + x_4 \le 80$$
$$2x_1 + 2x_3 + x_4 \le 60$$
$$x_j \ge 0, j = 1, 2, 3, 4$$

Solution:

Objective function is

$$\text{Minimize } z = -(\text{Maximize } (-z))$$

Consider the LP in standard form:

$$\text{Maximize } (-z) = -4x_1 + 3x_2 - 4x_3 + 6x_4 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 + s_1 = 80$$
$$3x_1 + 3x_2 + x_3 + x_4 + s_2 = 80$$
$$2x_1 + 2x_3 + x_4 + s_3 = 60$$
$$x_j \ge 0, j = 1, 2, 3, 4; \text{ and } s_1, s_2, s_3 \ge 0 \text{ are slack variables.}$$

Simplex table –1 (initial iteration)

		$C:\rightarrow$	-4	3	-4	6	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
0	s_1	80	1	2	2	4	1	0	0	80/4 = 20 ←
0	s_2	80	3	3	1	1	0	1	0	80/1 = 80
0	s_3	60	2	0	2	1	0	0	1	60/1 = 60
	z_j	0	0	0	0	0	0	0	0	
		$z_j - c_j$	4	-3	4	-6 ↑	0	0	0	

The basic variable s_1 corresponding to the first row should leave the basis and the variable x_4 corresponding to the column 4 should enter the basis.

The simplex table now reduces to the following:

Simplex table –2

		C:→	- 4	3	- 4	6	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
6	x_4	20	¼	½	½	1	¼	0	0	
0	s_2	60	11/4	5/2	½	0	- ¼	1	0	
0	s_3	40	7/4	- ½	3/2	0	- ¼	0	1	
	z_j	**120**	3/2	3	3	6	3/2	0	0	
		$z_j - c_j$	11/2	0	7	0	3/2	0	0	

The net evaluation $z_j - c_j$ is non-negative for every column j. Hence the basic feasible solution is an optimal basic feasible solution. The optimal value of the given objective function is –(120) and the corresponding optimal solution is $x_1 = 0, x_2 = 0, x_3 = 0$ and $x_4 = 6$.

Example 3: Use simplex method to

$$\textbf{Minimize } z = 4x_1 - 3x_2 + 4x_3 - 6x_4$$

Subject to the constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$$

$$3x_1 + 3x_2 + x_3 + x_4 \le 80$$

$$2x_1 + 2x_3 + x_4 \le 60$$

$$x_j \ge 0, j = 1, 2, 3, 4$$

Solution:

Consider the LP in standard form:

$$\text{Minimize} = 4x_1 - 3x_2 + 4x_3 - 6x_4 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 + s_1 = 80$$

$$3x_1 + 3x_2 + x_3 + x_4 + s_2 = 80$$

$$2x_1 + 2x_3 + x_4 + s_3 = 60$$

$x_j \ge 0, j = 1, 2, 3, 4$; and $s_1, s_2, s_3 \ge 0$ are slack variables.

Simplex table –1 (initial iteration)

		$C:\rightarrow$	4	–3	4	– 6	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
0	s_1	80	1	2	2	4	1	0	0	80/4 = 20
0	s_2	80	3	3	1	1	0	1	0	80/1 = 80
0	s_3	60	2	0	2	1	0	0	1	60/1 = 60
	z_j	0	0	0	0	0	0	0	0	
		$z_j - c_j$	– 4	3	– 4	6 ↑	0	0	0	

The basic variable s_1 corresponding to the first row should leave the basis and the variable x_4 corresponding to the column 4 should enter the basis.

The simplex table now reduces to the following:

Simplex table –2

		$C:\rightarrow$	4	– 3	4	– 6	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
– 6	x_4	20	¼	½	½	1	¼	0	0	
0	s_2	60	11/4	5/2	½	0	- ¼	1	0	
0	s_3	40	7/4	- ½	3/2	0	- ¼	0	1	
	z_j	**– 120**	– 3/2	– 3	– 3	– 6	– 3/2	0	0	
		$z_j - c_j$	– 11/2	0	– 7	0	– 3/2	0	0	

The net evaluation $z_j - c_j$ is non-positive for every column j. Hence the basic feasible solution is an optimal basic feasible solution. The optimal value of the given objective function is –120 and the corresponding optimal solution is $x_1 = 0, x_2 = 0, x_3 = 0$ and $x_4 = 6$.

Example 4. Two different products P_1 and P_2, can be manufactured by either of two different machines, M_1 and M_2. The unit processing time of either product on either machine is the same. The daily capacity of machine, M_1, is 200 units (of either P_1 or P_2, or a mixture of both) and the daily capacity of machine, M_2, is 250 units.

The shop supervisor wants to balance the production schedule of the two machines such that the total number of units produced on one machine is within 5 units of the number produced on the other. The profit per unit of P_1 is Rs. 10 and that of P_2 is Rs. 15.

Solve the problem using simplex method so as to get maximum profit.

Solution:

Let x_1 and x_2 be the number of units of production P_1 on machines M_1 and M_2 respectively. Let x_3 and x_4 be the number of units of production P_2 on machines M_1 and M_2 respectively. Then, as the profit from P_1 is Rs. 10 per unit and that of P_2 is Rs. 15 per unit, the profit is given by

$$Z = 10\,(x_1 + x_2) + 15\,(x_3 + x_4)$$

To find constraints;

Maximum capacity of M_1 is 200 $\Rightarrow x_1 + x_3 \le 200$ and $x_1 \le 200, x_2 \le 200 \Rightarrow x_1 + x_3 \le 200$

Maximum capacity of M_2 is 250 $\Rightarrow x_2 + x_4 \le 250$ and $x_2 \le 200, x_4 \le 400 \Rightarrow x_2 + x_4 \le 250$

Difference in total production on M_1 and total production on M_2 is at most 5

$$\Rightarrow |(x_1 + x_3) - (x_2 + x_4)| \le 5$$

$$\Rightarrow [\,(x_1 + x_3) - (x_2 + x_4)\,] \le 5 \quad \text{and} \quad -[(x_1 + x_3) - (x_2 + x_4)] \le 5$$

$$\Rightarrow x_1 + x_3 - x_2 - x_4 \le 5 \quad \text{and} \quad x_2 + x_4 - x_1 - x_3 \le 5$$

Non-negative restriction in production is

$$x_1, x_2, x_3, x_4 \ge 0$$

Thus the given problem in standard form is

Maximize $Z = 10\,x_1 + 10\,x_2 + 15\,x_3 + 15\,x_4 + 0\,s_1 + 0\,s_2 + 0\,s_3 + 0\,s_4$.

Subject to the constraints

$$x_1 + x_3 + s_1 = 200$$

$$x_2 + x_4 + s_2 = 250$$

$$x_1 - x_2 + x_3 - x_4 + s_3 = 5$$

$$-x_1 + x_2 - x_3 + x_4 + s_4 = 5$$

$$s_1, s_2, s_3, s_4, x_1, x_2, x_3, x_4 \ge 0$$

Simplex table – 1

		$C_j \rightarrow$	10	10	15	15	0	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	
0	s_1	200	1	0	1	0	1	0	0	0	200/1
0	s_2	250	0	1	0	1	0	1	0	0	---
0	s_3	5	1	-1	(1)	-1	0	0	1	0	← 5/1
0	s_4	5	-1	1	-1	1	0	0	0	1	---
	z_j	0	0	0	0	0	0	0	0	0	
		$z_j - c_j$	- 10	- 10	- 15 ↑	- 15	0	0	0	0	

x_3 should enter and s_3 should leave the basis. Performing the operations $R_1 \leftarrow R_1 - R_3$ and $R_4 \leftarrow R_4 + R_3$ we have,

Simplex table – 2

		$C_j \rightarrow$	10	10	15	15	0	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	
0	s_1	195	0	1	0	(1)	1	0	-1	0	← 195
0	s_2	250	0	1	0	1	0	1	0	0	250
15	x_3	5	1	-1	1	-1	0	0	1	0	----
0	s_4	10	0	0	0	0	0	0	0	1	----
	z_j	75	15	-15	15	-15	0	0	15	0	
		$z_j - c_j$	5	-25	0	-30 ↑	0	0	15	0	

x_4 should enter and s_1 should leave the basis. Performing the operations $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 + R_1$ we have,

Simplex table – 3

		$C:\rightarrow$	10	10	15	15	0	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	
15	x_4	195	0	1	0	1	1	0	-1	0	---
0	s_2	55	0	0	0	0	-1	1	1	0	55
15	x_3	200	1	0	1	0	1	0	0	0	---
0	s_4	10	0	0	0	0	0	0	0	1	---
	z_j	**5975**	15	15	15	15	30	0	-15	0	
		$z_j - c_j$	5	5	0	0	30	0	-15	0	

Thus s_3 should enter the basis and s_2 should leave the basis. Performing the operations $R_1 \leftarrow R_1 + R_2$, we get

Simplex table - 4

		$C:\rightarrow$	10	10	15	15	0	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	
15	x_4	250	0	1	0	1	0	1	0	0	
0	s_3	55	0	0	0	0	-1	1	1	0	
15	x_3	200	1	0	1	0	1	0	0	0	
0	s_4	10	0	0	0	0	0	0	0	1	
	z_j	**6750**	15	15	15	15	15	15	0	0	
		$z_j - c_j$	5	5	0	0	15	15	0	0	

Since $z_j - c_j$ is non-negative for every column j, the optimal solution is reached. The optimal value is 6750 and the optimal solution is $x_1 = 0, x_2 = 0, x_3 = 200$ and $x_4 = 250$. Thus to get maximum profit of Rs. 6750, firm has to produce only product P_1 in such a way that 200 units on machine M_1 and 250 units on machine M_2.

Example 5. Solve the following LPP using simplex method.

$$\textbf{Minimize } Z = x_1 + 2x_2 - 3x_3 - 2x_4$$

Subject to the constraints

$$x_1 + 2x_2 - 3x_3 + x_4 = 4;\quad x_1 + 2x_2 + x_3 + 2x_4 = 4;\quad x_1, x_2, x_3, x_4 \ge 0$$

Solution: Let $w = x_1 + 2x_2$. Then $w \ge 0$ (since $x_1, x_2 \ge 0$) and the given problem becomes

$$\text{Minimize } Z = w - 3x_3 - 2x_4$$

Subject to the constraints

$$w - 3x_3 + x_4 = 4 \qquad \ldots(1)$$

$$w + x_3 + 2x_4 = 4 \qquad \ldots(2)$$

$$w, x_3, x_4 \ge 0$$

This problem can also be written as

$$\text{Minimize } Z = w - 3x_3 - 2x_4$$

Subject to the constraints*

(2) – (1) gives;

$$4x_3 + x_4 = 0 \qquad \ldots(3)$$

2×(1) – (2) gives;

$$w - 7x_3 = 4 \qquad \ldots(4)$$

Simplex table – 1

		C:→	1	– 3	– 2	Ratio
c_B	y_B	x_B	w	x_3	x_4	
– 2	x_4	0	0	4	1	
1	w	4	1	– 7	0	
	z_j	4	1	– 15	– 2	
		$z_j - c_j$	0	– 12	0	

Since the net evaluation $z_j - c_j \ge 0$, for every column j, optimal solution is reached. The optimal value is $Z = 4$ and optimal solution is $w = 4$, $x_3 = 0$ and $x_4 = 0$. That is $w = x_1 + 2x_2 = 4$ is the optimal solution to the problem. This implies that the given problem contains an infinite number of solutions, they lie in the line segment $x_1 + 2x_2 = 4, x_1 \ge 0, x_2 \ge 0$.

Note: The solutions $x_1 = 0, x_2 = 2, x_3 = 0$ and $x_4 = 0$; $x_1 = 4, x_2 = 0, x_3 = 0$ and $x_4 = 0$ satisfy all the constrains and the optimal solution. The method of finding alternate solutions (if exists) dealt in next sections.

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* the variables w and x_4 serve the purpose of slack variables (observe?)

Example 6. Solve the following LPP using simplex method.

$$\text{Minimize } Z = \text{Max}\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$$

$$x_1, x_2, x_3 \geq 0.$$

Solution:

Let $y = \text{Max}\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$. Then y is non-negative and

$$y \geq |x_1 - x_2 + 3x_3| \text{ and } y \geq |-x_1 + 3x_2 - x_3|$$

$$\Rightarrow \quad y \geq x_1 - x_2 + 3x_3 \text{ and } y \geq -(x_1 - x_2 + 3x_3)$$

$$y \geq -x_1 + 3x_2 - x_3 \text{ and } y \geq -(-x_1 + 3x_2 - x_3)$$

$$\Rightarrow \quad -y + x_1 - x_2 + 3x_3 \leq 0 \text{ and } -y - x_1 + x_2 - 3x_3 \leq 0$$

$$-y - x_1 + 3x_2 - x_3 \leq 0 \text{ and } -y + x_1 - 3x_2 + x_3 \leq 0$$

Hence the given problem in standard form is

$$\text{Minimize } Z = y + 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to the constraints

$$-y + x_1 - x_2 + 3x_3 + s_1 = 0$$

$$-y - x_1 + x_2 - 3x_3 + s_2 = 0$$

$$-y - x_1 + 3x_2 - x_3 + s_3 = 0$$

$$-y + x_1 - 3x_2 + x_3 + s_4 = 0$$

$$y, x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$$

Simplex table –1

		C:→	1	0	0	0	0	0	0	0	Ratio
c_B	y_B	x_B	y	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
0	s_1	0	-1	1	-1	3	1	0	0	0	
0	s_2	0	-1	-1	1	-3	0	1	0	0	
0	s_3	0	-1	-1	3	-1	0	0	1	0	1
0	s_4	0	-1	1	-3	1	0	0	0	1	←0
	z_j	0	0	0	0	0	0	0	0	0	
		$z_j - c_j$	-1	0	0	0	0	0	0	0	

Since the problem is to minimize and the net evaluation $z_j - c_j \le 0$ for all j, the optimal solution is reached. Optimal value is 0 and the optimal solution is (0, 0, 0).

Note: - Some of the LP problems having constraints of the type LHS ≥ RHS ≥ 0 can also be solved using the same method; the following is one such example.

Example 7. Solve the following problem using simplex method (Do not use any artificial variables).

Minimize $Z = 3x_1 + 2x_2 + 3x_3$

Subject to the constraints

$$x_1 + 4x_2 + x_3 \ge 7$$

$$2x_1 + x_2 + x_4 \ge 10$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Solution: The given problem in standard form is

Minimize $Z = 3x_1 + 2x_2 + 3x_3 + 0x_4 + 0s_1 + 0s_2$

Subject to the constraints

$$x_1 + 4x_2 + x_3 - s_1 = 7$$

$$2x_1 + x_2 + x_4 - s_2 = 10$$

$$x_1, x_2, x_3, x_4, s_1, s_2 \ge 0$$

Taking x_3 and x_4 as a basic variables, we have

Simplex table – 1

		C:→	3	2	3	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	
3	x_3	7	1	4	1	0	-1	0	7/4
0	x_4	10	2	1	0	1	0	-1	10/1
	z_j	21	3	12	3	0	-3	0	
		$z_j - c_j$	0	10	0	0	-3	0	

Thus x_2 should enter and x_3 should leave the basis. Performing the operations $R_1 \leftarrow R_1/4$, $R_2 \leftarrow R_2 - R_1/4$, we get

Simplex table – 2

		C:→	3	2	3	0	0	0	Ratio
c_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	
2	x_2	7/4	¼	1	¼	0	- ¼	0	
0	x_4	33/4	7/4	0	- ¼	1	¼	-1	
	z_j	**14/4**	½	2	½	0	- ½	0	
		$z_j - c_j$	- 5/2	0	- 5/2	0	- ½	0	

Since net –evaluation $z_j - c_j$ is non-positive for every column j, the optimal solution is reached. The optimal value is 14/4 and the corresponding optimal solution is $x_1 = 0$, $x_2 = 7/4$, $x_3 = 0$ and $x_4 = 33/4$.

EXERCISES

I Write down the following LP problems in standard form:

(i) Max $z = 3x_1 + 2x_2 + 5x_3$

Subject to the constraints

$2x_1 - 3x_2 \le 3$

$x_1 + 2x_2 + 3x_3 \ge 5$

$3x_1 + 2x_3 \le 2$

$x_1, x_2 \ge 0$

(ii) Min $z = 2x_1 - x_2 + x_3$

Subject to the constraints

$x_1 + 3x_2 - x_3 \le 20$

$2x_1 - x_2 + x_3 \ge 12$

$x_1 - 4x_2 - 4x_3 \ge 2$

$x_1, x_2 \ge 0$, x_3 is unrestricted.

(iii) Maximize $z = 5x_1 - 6x_2 - 4x_3 + x_4$

Subject to the constraints

$x_1 + x_2 - x_3 = 5$

$x_2 - 3x_2 + x_4 \le -2$

$x_1 \ge 0$, $x_2 \le 0$, and, x_3 and x_4 are unrestricted.

II Use simplex method to solve the following LP problems

1. **Maximize $z = 3x_1 + 2x_2$**

Subject to the constraints

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0$$

2. Maximize $z = 2x_1 + 3x_2$

Subject to the constraints

$$x_1 + x_2 \leq 4$$
$$-x_1 + x_2 \leq 1$$
$$x_1 + 2x_2 \leq 5$$
$$x_1 \geq 0, x_2 \geq 0$$

3. Minimize $z = 8x_1 - 2x_2$

Subject to the constraints

$$-4x_1 + 2x_2 \leq 1$$
$$5x_1 - 4x_2 \leq 3$$
$$x_1 \geq 0, x_2 \geq 0$$

4. Use simplex method to Maximize $z = x_1 + x_2 + 3x_3$

Subject to the constraints

$$3x_1 + 2x_2 + x_3 \leq 3$$
$$2x_1 + x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0.$$

III Find the minimum and maximum values of the following LPP's using simplex method.

1. $z = x_1 + 5x_2 + 3x_3$

Subject to the constraints

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 \leq 4$$
$$x_1, x_2, x_3 \geq 0.$$

2. $z = 2x_1 + 3x_2 - 5x_3$

Subject to the constraints

$$x_1 + x_2 + x_3 = 7$$
$$5x_1 - x_2 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

IV A Company manufactures purses, shaving bags, and back packs. The construction of the three products includes leather and synthetic material, but leather seems to be the main limiting raw material. The production process requires two types of skilled labor; sewing and finishing. The following table gives the availability of the resources, their usage by the three products, and the profits per unit.

	Resource requirements per unit			
Resource	Purse	Bag	Backpack	Daily availability
Leather (mt^2)	2	1	3	42 mt^2
Sewing (hr)	2	1	2	40 hr
Finishing (hr)	1	0.5	1	45 hr
Selling price (Rs)	24	22	45	

Formulate the problem as a linear program and find its optimal solutions using simplex method.

V A pharmaceutical Company has 100 kg. of *A*, 180 kg. of *B* and 120 kg. of *C* available per month. They can use these material to make three basic pharmaceutical products namely 5–10–5, 5–5–10 and 20–5–10 where the numbers in each case represent the percentage by weight of *A*, *B* and *C* respectively in each of the products. The costs of these raw materials are given below:

Ingredient	Cost per kg. (Rs)
A	80
B	20
C	50
Inert ingredients	20

Selling price of these products are Rs. 40.50, Rs. 43 and Rs. 45 per kg. respectively. There is a capacity restriction of the company for the product 5–10–5, so as they cannot produce more than 30 kg per month. Determine how much of each of the products they should produce in order to maximize their monthly profit.

VI A computer manufacturing company manufacturers four types of computers. Each computer is first constructed in the assembly hall and is next sent to the testing place. The number of man-hours of labour required in for each purpose is as follows:

	Model 1	Model 2	Model 3	Model 4
Assembly	4	9	7	10
Testing	1	1	3	40
Profit from sale per unit (Rs.)	12	20	18	40

Because of limitation in capacity of the plant, no more than 6,000 man-hours can be expected in the assembly hall and 4,000 in the finishing shop in a month.

Assuming that raw materials are available in adequate supply and all computers produced can be sold, determine the quantities of each type of computers to be made for maximum profit of the company.

Answers: II. 1) 11 at (3, 1), (2) 8 at (4, 0) or (1, 2), 3) - 1 at (0, 1/2), 4) 3 at (0, 0, 3) V) 20,625 at (30, 1185, 0) VI) 1,340 at (20, 60, 30).

2.8 Big *M* Method (Penalty method)

The big M-method starts with the LP in its standard form. In this method, if any equation in the constraint in its standard form contains no slack variable, then we augment an artificial variable a_i to the equation. Such a variable then becomes a part of the starting basic solution. However, because artificial are extraneous to the LP model, we assign them a penalty in the objective function to force them to zero level at a later iteration of the simplex algorithm. The variable a_i is penalized in the objective function using $-Ma_i$ in the case of maximization and $+Ma_i$ in the case of minimization, where M is a largest positive number.

Example 1: Using simplex method solve the LPP

$$\text{Minimize } Z = 4x_1 + x_2$$

Subject to the constraints

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution: Consider the maximization problem in standard form,

$$\text{Max } (-Z) = -4x_1 - x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - s_1 = 6$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1, x_2\, s_1, s_2 \geq 0$$

where s_1 is a surplus variable and s_2 is a slack variable. Since the first and second equation does not contain slack variables, for the initial basic solution we should introduce artificial variables a_1 and a_2 respectively for these equations. The corresponding cost for this artificial variable is M (since the standard form is to maximize). Thus the standard form yields

$$\text{Max } (-Z) = -4x_1 - x_2 + 0s_1 + 0s_2 - Ma_1$$

Subject to the constraints

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1, x_2\, s_1, s_2, a_1 \geq 0$$

Simplex table - 1

		$C \rightarrow$	-4	-1	0	0	$-M$	$-M$	Ratio	
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
$-M$	a_1	3	3	1	0	0	1	0	$\frac{3}{3}=1$	←
$-M$	a_2	6	4	3	-1	0	0	1	$\frac{6}{4}=1.5$	
0	s_2	4	1	2	0	1	0	0	$\frac{4}{1}=4$	
	z_j	$-9M$	$-7M$	$-4M$	0	0	0	0		
		$z_j - c_j$	$-7M+4$ ↑	$-4M+1$	0	0	0	0		

Thus a_1 should leave and x_1 should enter the basis. Second iteration by applying the transformations $R_1 \rightarrow R_1/3$, $R_2 \rightarrow R_2 - (R_1/3) \times 4$, $R_3 \rightarrow R_3 - (R_1/3) \times 1$ we get

Simplex table - 2.

		$C \rightarrow$	-4	-1	0	0	$-M$	$-M$	Ratio	
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
-4	x_1	3/5	1	0	1/5	0	3/5	-1/5	3	
-1	x_2	6/5	0	1	-3/5	0	-4/5	3/5	-	
0	s_2	1	0	0	1	1	1	-1	1	←
	z_j	-18/5	-4	-1	-1/5	0	-8/5	2/5		
		$z_j - c_j$	0	0	-1/5 ↑	0	$-(8/5)+M$	$(2/5)+M$		

Therefore s_2 should leave and s_1 should enter. Thus applying $R_2 \rightarrow R_2 + \frac{R_3 \times 3}{5}$, $R_1 \rightarrow R_1 - \frac{R_3}{5}$, we get the next iteration as

Simplex table – 3

		$C \rightarrow$	-4	-1	0	0	$-M$	$-M$	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}
-4	x_1	$\frac{2}{5}$	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	
-1	x_2	$\frac{9}{5}$	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	
0	s_1	1	0	0	1	1	1	-1	
	z_j	$-\frac{17}{5}$	-4	1	0	$\frac{1}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	
		$z_j - c_j$	0	0	0	$\frac{1}{5}$	$-\frac{7}{5}+M$	$-\frac{8}{5}+M$	

The net evaluation $z_j - c_j$ is non-negative for every column j, hence the optimal solution is reached. The optimal value of the given problem is $-(17/5)$ and the optimal solution is $x_1 = 2/5$ and $x_2 = 9/5$.

Example 2. Using penalty method solve the LPP

Minimize $z = 3x_1 + 2x_2 + 3x_3$

Subject to constraints

$$x_1 + 4x_2 + x_3 \geq 7; \; 2x_1 + x_2 + x_4 \geq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: One way of solving the problem in Big M method by converting the minimization problem into maximization is studied in the above example. We now solve this problem by another method without converting into maximization problem. For such, the cost corresponding to the artificial variables is M (instead of $-M$) and the net evaluation should be most positive for the non-basic variables to be entered.

The given problem in standard form after introducing the slack/surplus variables and artificial variables in suitable places we get,

$$\text{Minimize } z = 3x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 + Ma_1 + Ma_2$$

Subject to constraints

$$x_1 + 4x_2 + x_3 - s_1 + a_1 = 7$$

$$2x_1 + x_2 + x_4 - s_2 + a_2 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2, a_1, a_2 \geq 0$$

Initial simplex table is

Simplex table – 1

		$C \rightarrow$	3	2	3	0	0	+M	+M	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
+M	a_1	7	1	4	1	-1	0	1	0	7/4	←
+M	a_2	10	2	1	1	0	-1	0	1	10/1	
	z_j	17M	3M	5M	2M	-M	-M	M	M		
		$z_j - c_j$	3M-3	5M-2	2M-3	-M	-M	0	0		
				↑ Most + ve.							

Thus a_1 should leave and x_2 should enter the basis. Now performing the operation

$R_1 \leftarrow \frac{R_1}{4}$, $\quad R_2 \leftarrow R_2 - \frac{R_1}{4} \times 1$, we get the following iteration.

Simplex table – 2

		$C \rightarrow$	3	2	3	0	0	+M	+M	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
2	x_2	7/4	¼	1	¼	-¼	0	¼	0	7	
M	a_2	33/4	7/4	0	¾	¼	-1	-¼	1	33/7	←
	z_j	$\frac{7}{2} + \frac{33M}{4}$	[redacted]	2	$\frac{3M}{4} - \frac{1}{2}$	$\frac{M}{4} - \frac{1}{2}$	-M	$\frac{-M}{4} + \frac{1}{2}$	M		
		$z_j - c_j$	[redacted]	0	$\frac{3M}{4} - \frac{7}{2}$	$\frac{M}{4} - \frac{1}{2}$	-M	$\frac{-5M}{4} + \frac{1}{2}$	0		
			↑								

Thus a_2 should leave x_2 should enter the basis. For this performing the operation $R_2 \leftarrow \frac{4R_2}{7}$, $R_1 \leftarrow R_1 - (R_2/7)$ we get the following table.

		$C \rightarrow$	3	2	3	0	0	$+M$	$+M$	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}
2	x_2	4/7	0	1	1/7	-2/7	1/7	2/7	-1/7	
3	x_1	33/7	1	0	3/7	1/7	-4/7	-1/7	4/7	
	z_j	**117/7**	3	2	11/7	-1/7	-10/7	1/7	1/7	
		$z_j - c_j$	0	0	-10/7	-1/7	-10/7	$\frac{1}{7} - M$	$\frac{10}{7} - M$	
			≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	≤ 0	

Since the net evaluation $z_j - c_j$ is non-positive for all the columns the optimal solution is reached. Optimal solution is $x_1 = 33/7$, $x_2 = 4/7$ and $x_3 = 0$. The corresponding optimal value is $Z = 117/7$.

Example 3: Solve the following LPP using big M method

$$\textbf{Maximize } Z = 2x_1 + 5x_2$$

Subject to the constraints

$$3x_1 + 2x_2 \geq 6;\ \ 2x_1 + x_2 \leq 2;\ \ x_1, x_2 \geq 0$$

Solution: Converting the given problem into standard form by introducing slack/surplus variables and suitable artificial variables in suitable places, we get

$$\text{Maximize } Z = 2x_1 + 5x_2 + 0s_1 + 0s_2 - Ma_1.$$

Subject to the constraints

$$3x_1 + 2x_2 - s_1 + a_1 = 6$$

$$2x_1 + x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Simplex table – 1

		$C \rightarrow$	2	5	0	0	-M	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	x_{Bi}/y_{ij}
-M	a_1	6	3	2	-1	0	1	2
0	s_2	2	2	1	0	1	0	1 ←

z_j	$-6M$	$-3M$	$-2M$	M	0	$-M$
	$z_j - c_j$	$-3M-2$	$-2M-5$	M	0	0
		↑				

Thus the non-basic variable x_1 should enter and s_2 should leave the basis. Performing the operations $R_2 \to (R_2/2)$ and $R_1 \to R_1 - (R_2/2) \times 3$ we get

Simplex table – 2

		$C \to$	2	5	0	0	$-M$	Ratio	
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	x_{Bi}/y_{ij}	
$-M$	a_1	3	0	1/2	-1	- 3/2	1	6	
2	x_1	1	1	1/2	0	1/2	0	2	←
	z_j	$2-3M$	2	1[redacted]	M	$1+\frac{3M}{2}$	$-M$		
		$z_j - c_j$	0	[redacted]	M	$1+\frac{3M}{2}$	0		
				↑					

Thus the variables x_1 should leave and x_2 should enter. Performing the operations $R_2 \to 2R_2$ and $R_1 \to R_1 - R_2$, we get

Simplex table – 3

		$C \to$	2	5	0	0	$-M$	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	x_{Bi}/y_{ij}
$-M$	a_1	2	-1	0	-1	-2	1	
5	x_2	2	2	1	0	1	0	
	z_j	$7-2M$	$7+M$	5	M	$6+2M$	$-M$	
		$z_j - c_j$	$5+M$	0	M	$6+2M$	0	
			≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	

Since the net evaluation is non-negative for all the columns and the artificial variable present in the basis. Hence the given problem has no feasible solution*.

Observations on Big M method:

There are two observations regarding the use of the big M- method.

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* see observations on big M method

1. If the given LP does not have a feasible solution, then the final simplex iteration will include at least one artificial variable at a positive level (i.e. the artificial variable still present in the basic solution).
2. The value of M must be big. But for a given problem how large is "sufficiently large" is an open question. Specifically, M should be large enough to act as a penalty, but it should not be too large to impair the accuracy of the simplex computations.

The observation 2 shows that the unqualified theoretical requirement that requires M to be selected "very large" can lead to serious round off error. Due to this reason the big M-method is never implemented in commercial LP codes. The two-phase method is an improvement in this direction.

Before going to study next method, we observe the following example.

Example 4. Solve the following system of equations using simplex method.

$$x_1 + x_2 = 5$$
$$2x_1 - 3x_2 = 5.$$

Solution: Assuming the trivial objective function

$$\text{Max } Z = 0x_1 + 0x_2 + 0a_1 + 0a_2$$

Subject to the constraints

$$x_1 + x_2 + a_1 = 5$$
$$2x_2 - 3x_2 + a_2 = 5$$

First iteration

Simplex table – 1.

		$C\rightarrow$	0	0	$-M$	$-M$	Ratio
C_B	y_B	x_B	x_1	x_2	a_1	a_1	x_{Bi}/y_{ij}
$-M$	a_1	5	1	1	1	0	5
$-M$	a_2	5	2	-3	0	1	5/2
	z_j	$-10M$	$-3M$	$2M$	$-M$	$-M$	
		$z_j - c_j$	$-3M-2$	$2M-5$	0	0	

Thus a_2 should leave and x_1 should enter the basis. Performing operations $R_2 \leftarrow R_2/2$ and $R_1 \leftarrow R_1 - R_2/2$, we get

Simplex table – 2

		$C\rightarrow$	0	0	$-M$	$-M$	Ratio
C_B	y_B	x_B	x_1	x_2	a_1	a_1	x_{Bi}/y_{ij}

-M	a_1	5/2	0	5/2	1	-½	1
0	x_1	5/2	1	-3/2	0	½	-
	z_j	$\frac{-5M}{2}$	2	[redacted]	-M	$\frac{-M}{2}+1$	
		z_j-c_j	0	[redacted]	0	$\frac{M}{2}+1$	

Thus x_2 should enter and a_1 should leave the basis. Performing the operations $R_1 \leftarrow 2R_1/5$ and $R_2 \leftarrow R_2+(3R_1/5)$, we get

Simplex table – 3

		$C \rightarrow$	0	0	-M	-M	Ratio
C_B	y_B	x_B	x_1	x_2	a_1	a_1	x_{Bi}/y_{ij}
0	x_2	1	0	1	2/5	-1/5	
0	x_1	4	1	0	3/5	1/5	
	z_j	0	2	0	0	0	
		z_j-c_j	0	0	M	M	

Since $z_j - c_j \geq 0$, for all the columns optimal solution is reached. The optimal value of the function is $x_1 = 4$ and $x_2 = 1$.

Note: From the above example it is clear that the simplex iteration looks only corner point that are the intersection of the lines corresponding to the given constrains. Thus to get the corner points we need not bother about the objective function. This idea is used while developing the following algorithm.

2.9 Two-Phase Method

The two-phase method is designed to alleviate the problem by eliminating the constant M altogether. As a name indicates, this method solves the problem in two phases: Phase I attempts to find a basic feasible solution. If such a basic solution is found in phase I, then phase II uses it to solve the original problem. These phases are;

Phase I: Consider the problem in standard LP form. Add the necessary artificial variables to the constraints similar to big M-method. Start with the simplex table-1 by assigning zero cost vector for all the variables (including slack and surplus), except for the artificial variables in the objective function. Assign a cost of -1 for each artificial variable. Find a basic solution of the resulting equations that minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process. Otherwise, we move to Phase II.

Phase II: Use the feasible solution obtained in Phase I as a starting basic feasible solution for the original problem by assigning actual cost vector to it.

Example 5: Solve the following LP problem using two-phase simplex method

Minimize $Z = 4x_1 + x_2$

Subject to the constraints

$$3x_1 + x_2 = 3;\ 4x_1 + 3x_2 \geq 6;\ x_1 + 2x_2 \leq 4;\ x_1, x_2 \geq 0$$

Solution: The given problem in standard form with artificial variables is

Maximize $(-Z) = -4x_1 - x_2 + 0s_1 + 0s_2$

Subject to the constraints

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

Phase 1:

Taking zero cost vector for all variables except artificial variables and a cost –1 for the artificial variables, we have

Simplex table –1

		$C\rightarrow$	0	0	0	0	-1	-1	Ratio	
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
-1	a_1	3	3	1	0	0	1	0	1	←
-1	a_2	6	4	3	-1	0	0	1	1.5	
0	s_2	4	1	2	0	1	0	0	4	
	z_j	-9	-7	-4	1	0	-1	-1		
		$z_j - c_j$	-7	-4	1	0	0	0		
			↑							

Thus, x_1 should enter and a_1 should leave. Performing the operations $R_1 \rightarrow R_1/3$, $R_2 \rightarrow R_2 - \frac{R_2}{3}\times 4$ and $R_3 \rightarrow R_3 - R_1/3$; we get

Simplex table – 2

		$C\rightarrow$	0	0	0	0	-1	-1	Ratio	
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2	x_{Bi}/y_{ij}	
0	x_1	1	1	1/3	0	0	1/3	0	3	
-1	a_2	2	0	5/3	-1	0	-4/3	1	6/5	←

0	s_2	3	0	5/3	0	1	-1/3	0	9/5
	z_j	-2	0	-5/3	1	0	4/3	-1	
		$z_j - c_j$	0	-5/3	1	0	7/3	0	
				↑					

Thus x_2 should enter and a_2 should leave. Performing the operations $R_2 \to \frac{3R_2}{5}$, $R_1 \to R_1 - \frac{3}{5} R_2 \times \frac{1}{3}$ and $R_3 \to R_3 - R_2$ we get

Simplex table – 3

		$C \to$	0	0	0	0	-1	-1
C_B	y_B	x_B	x_1	x_2	s_1	s_2	a_1	a_2
0	x_1	3/5	1	0	1/5	0	3/5	-1/5
0	x_2	6/5	0	1	-3/5	0	-4/5	3/5
0	s_2	1	0	0	1	1	1	-1
	z_j	**0**	0	0	0	0	0	0
		$z_j - c_j$	0	0	0	0	1	1

Since the net evaluation is non-negative for all the columns and the value of the objective function is 0. The phase 1 terminates and the solution obtained can be taken as a basic feasible solution to the next phase.

Phase (ii)

Now assigning actual costs to the corresponding variables and neglecting the artificial variables we get

Simplex table – 4

		$C \to$	-4	-1	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	x_{Bi}/y_{ij}
-4	x_1	3/5	1	0	1/5	0	3
-1	x_2	6/5	0	1	-3/5	0	-
0	s_2	1	0	0	1	1	1
	z_j	**-18/5**	-4	-1	-1/5	0	
		$z_j - c_j$	0	0	-1/5	0	

Thus s_1 should leave and s_2 should leave enter. Performing the operations $R_1 \leftarrow R_1 - (1/5)R_3$ and $R_2 \leftarrow R_2 + (3/5)R_1$.

		$C \rightarrow$	-4	-1	0	0
C_B	y_B	x_B	x_1	x_2	s_1	s_2
-4	x_1	2/5	1	0	0	-1/5
-1	x_2	9/5	0	1	0	3/5
0	s_2	1	0	0	1	1
	z_j	-17/5	-4	-1	-1/5	0
		$z_j - c_j$	0	0	-1/5	0
			≥ 0	≥ 0	≥ 0	≥ 0

Since the net evaluation is non-negative, the optimal solution is reached. The optimal value is 17/5 (since the problem is to minimize) and the optimal solution is $x_1 = 2/5$ and $x_2 = 9/5$.

Example 6. Minimize $z = 3x_1 + 2x_2 + 3x_3$

Subject to the constraints

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 4$$
$$x_1, x_2 \geq 0.$$

Solution: The standard form of the given LP problem is

$$\text{Minimize } z = 3x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2$$

Subject to the constraints

$$2x_1 + x_2 + x_3 + s_1 = 2$$
$$3x_1 + 4x_2 + 2x_3 - s_2 + a_1 = 4$$
$$x_1, x_2, s_1, s_2, a_1 \geq 0.$$

Phase 1:

Taking zero cost vectors for all variables except artificial variables and a cost −1 for the artificial variables, we have

Simplex table –1

		$C \rightarrow$	0	0	0	0	0	-1	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	a_1	x_{Bi}/y_{ij}	
0	s_1	2	2	1	1	1	0	0	2/1=2	
-1	a_1	4	3	4	2	0	-1	1	4/4=1	←
	z_j	-4	-3	-4	-2	0	1	-1		
		$z_j - c_j$	-3	-4	-2	0	1	0		
				↑						

Thus a_1 should leave and x_3 should enter. Performing the operations $R_1 \leftarrow R_1 - \frac{R_2}{4} \times 1$ and $R_2 \rightarrow R_2/4$ we get,

Simplex table – 2

		$C \rightarrow$	0	0	0	0	0	- 1	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	a_1	x_{Bi}/y_{ij}
0	s_1	1	5/4	0	½	1	¼	-1/4	
0	x_2	1	¾	1	½	0	-1/4	¼	
	z_j	**0**	0	0	0	0	0	0	
		$z_j - c_j$	0	0	0	0	0	1	

Since the net evaluation is non-negative for all the columns and the value of the objective function is 0. The phase 1 terminates and the solution obtained can be taken as a basic feasible solution to the next phase.

Phase (ii)

Now assigning a actual cost to the variables and neglecting the artificial variables we get ,

		$C \rightarrow$	3	2	3	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	s_1	s_2	x_{Bi}/y_{ij}
0	s_1	1	5/4	0	½	1	¼	
2	x_2	1	¾	1	½	0	-1/4	
	z_j	2	3/2	2	1	0	-1/2	
		$z_j - c_j$	-3/2	0	-2	0	-1/2	

Since the net evaluation for all the columns is non-positive, the optimal solution is reached. $\therefore$ min $Z = 2, x_1 = 0, x_2 = 1, x_3 = 0$.

Example 7. Using simplex method solve:

$$\text{Minimize } Z = 2x_1 - 4x_2 + 3x_3$$

Subject to the constraints

$$5x_1 - 6x_2 + 2x_3 \geq 5$$
$$-x_1 + 3x_2 + 5x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Show that the inequalities can be modified to a set of equations that requires the use of single artificial variable only (instead of two).

Solution: Consider the constraints

$$5x_1 - 6x_2 + 2x_3 \geq 5$$

$$-x_1 + 3x_2 + 5x_3 \geq 8$$

Multiplying second constraint by 5 and adding with the first equation, we get

$$9x_2 + 27x_3 \geq 45$$

Now the given problem is equivalent to the problem

Minimize $Z = 2x_1 - 4x_2 + 3x_3$

Subject to the constraints

$$5x_1 - 6x_2 + 2x_3 \geq 5$$

$$9x_2 + 27x_3 \geq 45$$

Now in standard form

Minimize $Z = 2x_1 - 4x_2 + 3x_3 + 0s_1 + 0s_2$

Subject to the constraints

$$x_1 - (6/5)x_2 + (2/5)x_3 - s_1 = 1$$

$$9x_2 + 27x_3 - s_1 = 45$$

The variables x_1 and a_1 can be taken as a initial basic variable. We leave here and exercise to solve the problem using simplex method.

2.10 Special cases in Simplex Method Application

There are four special cases that arise in the application of simplex method namely

1. Degeneracy
2. Alternative Optima
3. Unbounded solution
4. Infeasible solution

Degeneracy: While determining a pivotal element in a simplex table in any stage there is a tie for the minimum ration occur, that is, there are more rows corresponding the column for which the net evaluation $z_j - c_j$ is minimum have the same minimum ratio. When this happens, one or more of the basic variables will be zero in the next iteration. In this case, the new solution is degenerate. This means that the subsequent iteration will not improve in the value of the objective function and the computation will never terminate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solution. This concept is known as cycling.

A computation procedure to avoid cycling at any stage consists of the following steps:

Step 1. Let the pivotal element lies in the j^{th} column. Let r_1, r_2 be the rows for which there is a tie. Let $k = 1$.

Step 2. Choose k^{th} basic vector (the column for which the k^{th} row entry is 1 and all other is zero).

Step 3. Choose the pivotal y_{ij}, where i is such that, $\frac{y_{ik}}{y_{ir}} = \min_{l}\left\{\frac{y_{lk}}{y_{lr}}, y_{lr} > 0\right\}$ for all those rows l for which there is a tie.

Step 4. If there is a tie in the choice of i in the above step 3, then $k \leftarrow k + 1$, go to step 2.Otherwise continue the simplex table to obtain next feasible solution.

Example 8. Solve the following L.P.P.

$$\textbf{Max } Z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4$$

Subject to the constraints:

$$\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \le 0$$

$$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \le 0$$

$$x_3 \le 1, x_1, x_2, x_3, x_4 \ge 0.$$

Solution: The given problem in standard form is

$$\text{Max } Z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4$$

Subject to the constraints:

$$\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + s_1 = 0$$

$$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + s_2 = 0$$

$$x_3 + s_3 = 1$$

$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \ge 0.$$

Simplex table -1

		$C \rightarrow$	¾	-150	1/50	-6	0	0	0	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	x_{Bi}/y_{ij}	
0	s_1	0	¼	-60	-1/25	9	1	0	0	0	←
0	s_2	0	½	-90	-1/50	3	0	1	0	0	←

0	s_3	1	0	0	1	0	0	0	1	--
	z_j	0	0	0	1	0	0	0	1	
		$z_j - c_j$	-3/4	150	49/50	6	0	0	1	

↑

In the above table we observe that the minimum ratio is same for two rows namely first and second row. This is the case of degeneracy, in such cases we have to choose the first column of the identity matrix corresponding to the basic variables and compute the ratio for these elements. For example in the above case we take the column s_1 which is the first column of the identity matrix. Computing the ratio for this column we get 1/(¼) = 4 and 0/(½) = 0. The minimum of these is zero, which corresponds to the row 2. Hence we choose the row 2 (if there is a tie again to choose the row consider the next column of the identity matrix and repeat the process).

		$C \rightarrow$	¾	-150	1/50	-6	0	0	0	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	x_{Bi}/y_{ij}	
0	s_1	0	¼	-60	-1/25	9	1	0	0	4	
0	s_2	0	½	-90	-1/50	-5	0	1	0	0	←
0	s_3	1	0	0	1	0	0	0	1		
	z_j	0	0	0	1	0	0	0	1		
		$z_j - c_j$	-3/4	150	49/50	6	0	0	1		

↑

Now x_1 should enter and s_2 should leave. Performing the operations $R_1 \rightarrow R_1 - 2R_2 \times \frac{1}{4}$, $R_2 \rightarrow R_2/½$ and $R_3 \rightarrow R_3 - 2R_2 \times 0$, we get

Simplex table - 2

		$C \rightarrow$	¾	-150	1/50	-6	0	0	0	Ratio	
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	x_{Bi}/y_{ij}	
0	s_1	0	0	-15	-3/100	15/2	1	- 1/2	0	-	
¾	x_1	0	1	-180	-1/25	6	0	2	0	-	
0	s_2	1	0	0	1	0	0	0	1	1	←
	z_j	0	¾	-135	-3/100	9/2	0	3/2	0		

$z_j - c_j$	0	15	-1/20	21/2	0	3/2	0

↑

Thus, s_2 should leave and x_3 should enter. Performing the operations

$R_3 \leftarrow R_3 +1$, $R_2 \rightarrow R_2 - \frac{R_3}{1} \times \frac{-1}{25} = R_2 + \frac{R_3}{25}$ and $R_1 \leftarrow R_1 + \frac{3R_3}{100}$ we get

Simplex table – 3

		$C \rightarrow$	¾	-150	1/50	-6	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1	s_2	S_3	x_{Bi}/y_{ij}
0	s_1	3/100	0	-15	0	15/2	1	-1/2	3/100	1/250
¾	x_1	1/25	1	-180	-1/25	6	0	2	1/25	1/150
1/50	x_3	1	0	0	1	0	0	0	1	-
	z_j	1/20	¾	-135	1/50	9/2	0	3/2	1/20	
		$z_j - c_j$	0	15	0	21/2	0	3/2	1/20	

The net evaluation for every column is non-negative therefore optimal solution is reached. The optimal value = 1/20, and the optimal solution is x_1 = 1/25, x_2 = 0, x_3 = 1 and x_4 = 0

Alternative Optima: If the net evaluation in the final iteration corresponding to non-basic variable is zero, then there exists an alternative basic solution for the problem. This can be obtained by further continuing the table bringing the corresponding non-basic variable into basic by replacing any of the slack variables with it. More over if an L.P.P. contains two solutions any X_1 and X_2, then $\lambda X_1 + (1 - \lambda) X_2$ is also a solution, for every λ, $0 \le \lambda \le 1$. Hence there are infinite numbers of solutions for the problem.

Example 9. Use simplex method to solve the following L.P.P.

Max $Z = 4x_1 + 10x$

Subject to the constraints:

$2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \ge 0$. Is the solution unique? If not, find all its solutions.

Solution: The given problem in standard form is

Max $Z = 4x_1 + 10x$

Subject to the constraints:

$$2x_1 + x_2 + s_1 = 50$$

$$2x_1 + 5x_2 + s_2 = 100$$
$$2x_1 + 3x_2 + s_3 = 90$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Phase I: since artificial variables not present phase I is inadmissible

Phase II:

Simplex table – 1

		$C\rightarrow$	4	10	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	x_{Bi}/y_{ij}
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	5	0	1	0	20 ←
0	s_3	90	2	-3	0	0	1	-
	z_j	0	0	0	0	0	0	
		$z_j - c_j$	-4	-10	0	0	0	
				↑				

Thus, x_1 should enter and s_2 should leave. Per forming the operations $R_2 \rightarrow R_2/5$, $R_1 \leftarrow R_1 - \frac{R_2}{5}\times 1$ and $R_3 \leftarrow R_3 - \frac{R_2}{5}\times(-3)$ we have

Simplex table – 2

		$C\rightarrow$	4	10	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	x_{Bi}/y_{ij}
0	s_1	30	8/5	0	1	-1/5	0	
10	x_2	20	2/5	1	0	1/5	0	
0	s_3	150	16/5	0	0	3/5	1	
	z_j	200	4	10	0	2	0	
		$z_j - c_j$	0	0	0	2	0	

Since the net evaluation is non-negative for every column, the optimal solution is reached. Optimal value is 200 and optimal solution is $x_1 = 0$, $x_2 = 20$. Further we observe that solution is not unique as x_1 is a non –basic variable and net evaluation $z_j - c_j$ for x_2 is zero (not positive). x_1 can enter the basic and one of the slack say s_1 (as $z_j - c_j =$ 0)can leave (similarly try to get other solution by entering x_1 and leaving s_3) . To get the new optimal solution perform the operations

$R_1 \to \frac{5R_1}{8}, R_2 \to R_2 - \frac{2R_1}{8}$ and $R_3 \to R_3 - \frac{16R_1}{8} = R_3 - 2R_1$

		C	4	10	0	0	0	Ratio
C_B	y_B	x_B	x_1	x_2	s_1	s_2	s_3	x_{Bi}/y_{ij}
4	x_1	75/4	1	0	5/8	-1/8	0	
10	x_2	25/2	0	1	-1/4	¼	0	
0	s_3	90	0	0	-2	1	1	
	z_j	200	4	10	0	2	0	
		$z_j - c_j$	0	0	0	2	0	

Optimal value = 200 Optimal solution is $x'_1 = 75/4$, $x'_2 = 25/2$. Since there are two solutions. All the other solutions are the points on the line segment joining these two solution points. That is the solution $X = \alpha X_1 + (1 - \alpha) X_2$, for every α such that $0 \le \alpha \le 1$, where $X_1 = (x_1, x_2)^T$ and $X_2 = (x'_1, x'_2)^T$.

Unbounded Solution:

If the net evaluation $z_j - c_j$ is most negative for the j^{th} column, $y_{ij} \le 0$ for all i, then it is an indication of unbounded solution.

Example 10. Consider the LP

$$\text{Max } Z = 20x_1 + 10x_2 + x_3$$

Subject to the constraints

$$3x_1 - 3x_2 + 5x_3 \le 50, \quad x_1 + x_3 \le 10$$
$$x_1 - x_2 + 4x_3 \le 20, \quad x_1, x_2, x_3 \ge 0.$$

By inspecting the constraints, determine the direction in which the solution space is unbounded. Without further computations, what can you conclude regarding the optimum objective value?

Solution: The given problem in standard form is

$$\text{Max } Z = 20x_1 + 10x_2 + x_3$$

Subject to the constraints

$$3x_1 - 3x_2 + 5x_3 + s_1 = 50$$
$$x_1 + x_3 + s_2 = 10$$
$$x_1 - x_2 + 4x_3 + s_3 = 20$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$$